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Stochastic resonance in the FitzHugh-Nagumo model from a dynamic mutual information point of view

T. Munakata[1,a] and M. Kamiyabu[2]

[1] Department of Applied Mathematics and Physics, Kyoto University, Kyoto 606-8501, Japan
[2] Software Development Department, Communication Systems Group, Sharp Corporation, Higashi-Hiroshima 739-0192, Japan

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Abstract. Stochastic resonance(SR) in a FitzHugh-Nagumo neuron model is investigated based on a dynamic mutual information (DMI) between the input and the corresponding output signals. The DMI is expressed in terms of the (cross)power spectra of the input and output time series. Both stochastic-periodic and aperiodic SR are treated based on the DMI and our results are in good accord with the SR measured by the signal to noise ratio(SNR) for the case of the stochastic-periodic input and the power norm for the case of the aperiodic input.

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1 Introduction

Stochastic resonance(SR), which denotes the effect in which the transmission of input signal information is enhanced by adding random noise in some nonlinear systems, has gathered much attention in the field of stochastic systems [1,2]. SR was first reported for a bistable system under weak periodic perturbation, a model proposed to discuss the glacial period of earth [3]. Since then it is extended to many nonlinear systems such as a simple threshold system [4] and a monostable excitable system like neurons [5], to mention a few.

SR was originally measured by a signal-to-noise ratio(SNR) and was characterized as a maximum of the SNR at non-zero noise intensity [3]. Here the SNR is defined as the ratio of the weight of the δ peak of the power spectrum of the output signal at the characteristic input frequency ω_0 to the background intensity of the power spectrum at the same frequency [3,6]. When the input signal has not such a characteristic frequency, one can not resort to the SNR just defined above and other measures for SR, such as the power norm which quantifies similarity between the input and output signals, have been proposed and applied in the field of excitable systems [7].

In view of the significance of these stochastic systems as information processing devices, mutual information (MI) [8–10] between the input and output(response) signals has also been playing important roles. The purpose of this report is to study SR in a FitzHugh-Nagumo (FHN) neuron model from a unified viewpoint of dynamic mutual information (DMI).

First we introduce in Section 2 the DMI which is expressed concisely, under a Gaussian approximation, in terms of the auto- and cross-correlation functions of both input and output signals. This DMI is then applied to a FitzHugh-Nagumo (FHN) neuron in Section 3 to discuss both periodic and aperiodic SR from a unified viewpoint. Final section is devoted to a summary.

2 Dynamic mutual information (DMI)

MI between the input $\boldsymbol{I} = (i_1, ..., i_N)$ and output $\boldsymbol{O} = (o_1, ..., o_N)$ signals $M(\boldsymbol{I} : \boldsymbol{O})$ can be simply represented as [8]

$$\begin{aligned} M(\boldsymbol{I} : \boldsymbol{O}) &\equiv H(\boldsymbol{I}) - H(\boldsymbol{I}|\boldsymbol{O}) = H(\boldsymbol{O}) - H(\boldsymbol{O}|\boldsymbol{I}) \\ &= H(\boldsymbol{I}) + H(\boldsymbol{O}) - H(\boldsymbol{I}, \boldsymbol{O}), \end{aligned} \quad (1)$$

where $H(\boldsymbol{I})$, the Shannon entropy, quantifies uncertainty in input signals and $H(\boldsymbol{I}|\boldsymbol{O})$, the conditional entropy, quantifies (remaining) uncertainty in input signals once output signals are given [8]. $H(\boldsymbol{O})$, $H(\boldsymbol{O}|\boldsymbol{I})$ and $H(\boldsymbol{I}, \boldsymbol{O})$ have similar meaning and the equalities in (1) are simply derived from the definitions for these quantities to be given below. $M(\boldsymbol{I} : \boldsymbol{O})$ thus measures information flow and it is

[a] e-mail: munakata@amp.i.kyoto-u.ac.jp

expected that this quantity would be usefully applied for SR especially for neuronal systems which are supposed to transmit information efficiently.

We denote the distribution function of input(output) signals $\boldsymbol{I}(\boldsymbol{O})$ by $p_I(\boldsymbol{I})(p_O(\boldsymbol{O}))$ and introduce the conditional distribution function $p_{I|O}(\boldsymbol{I}|\boldsymbol{O})$ of $\boldsymbol{I}$ when $\boldsymbol{O}$ is given. In terms of these functions we define [8]

$$H(\boldsymbol{I}) = -\sum_{\boldsymbol{I}} p_I(\boldsymbol{I}) \ln p_I(\boldsymbol{I}),$$
$$H(\boldsymbol{I}|\boldsymbol{O}) = -\sum_{\boldsymbol{O}} p_O(\boldsymbol{O}) \sum_{\boldsymbol{I}} p_{I|O}(\boldsymbol{I}|\boldsymbol{O}) \ln p_{I|O}(\boldsymbol{I}|\boldsymbol{O}). \quad (2)$$

Similarly $H(\boldsymbol{I},\boldsymbol{O})$ is expressed in terms of the joint distribution $p_{I,O}(\boldsymbol{I},\boldsymbol{O})$ as

$$H(\boldsymbol{I},\boldsymbol{O}) = -\sum_{\boldsymbol{O}}\sum_{\boldsymbol{I}} p_{I,O}(\boldsymbol{I},\boldsymbol{O}) \ln p_{I,O}(\boldsymbol{I},\boldsymbol{O}). \quad (3)$$

Recenly we classified an information processing system into systems with and without memory [10]. When o_n in $\boldsymbol{O} = (o_1,..,o_n,..,o_N)$ depends on only i_n in $\boldsymbol{I} = (i_1,..,i_N)$, the system is without memory and we studied MI based on an approximation of one-body or static MI for a simple threshold system [10]. When system dynamics is described by a differential equation, such as the FitzHugh-Nagumo (FHN) equation, see Section 3, the system is in general with memory in the sense that o_n depends on $i_n, i_{n-1},..$ and one must take full dynamical information in $\boldsymbol{I}$ and/or $\boldsymbol{O}$ into account.

In numerically calculating $M(\boldsymbol{I}:\boldsymbol{O})$ one must first obtain the probability distribution functions in (2) and (3) and this is a formidable task. This is because we need generally a lot of data points in order to calculate a smooth distribution function from the data. Neiman et al. [11] calculated numerically some entropic quantities in their studies on stochastic Schmitt trigger. Bulsara and Zador [9] used the interspike interval (ISI) distribution function combined with Monte Carlo calculations, which tactfully reduced numerical calculations. Heneghan et al. [12] also made use of the rate of information transfer under the Gaussian approximation for the input noise in their study of aperiodic SR. $M(\boldsymbol{I}:\boldsymbol{O})$ for $N = 1$ is widely used for simple threshold systems and also for continuous systems after quantization of signals [13].

Difficulties associated with numerical calculations of $M(\boldsymbol{I}:\boldsymbol{O})$ are greatly reduced once one employs a Gaussian approximation for both the input and output processes and the remaining part of this section is devoted to a derivation of the formula (12) below, which will be used in Sect.3. The result (12) is nontrivial, not widely known, and useful (as shown in Sect. 3), so that the derivation thereof is included here also to make this paper self-contained. The calculations are along with a similar line in partial integration in path integrals for linear quantum systems [14].

Denoting by Δ the sampling time, we take N sample points $\boldsymbol{I} \equiv \{i_1 = i(t=\Delta),...,i_N = i(t=N\Delta)\}$ from an input stochastic process $i(t)$. With the covariance matrix A, the element of which is expressed in terms of the stationary correlation function $c_{I,I}(t)$ as $A_{j,k} = \langle i_j i_k\rangle \equiv c_{I,I}(|j-k|\Delta)$, the distribution function $p_I(\boldsymbol{I})$ takes the form

$$p_I(\boldsymbol{I}) = (2\pi)^{-N/2}|A|^{-1/2}\exp[-(1/2)\boldsymbol{I}^T\cdot A^{-1}\cdot\boldsymbol{I}], \quad (4)$$

where $|A|$ is a determinant of A and T on the column vector $\boldsymbol{I}$ denotes the transpose operation.

With use of an orthogonal transformation to the matrix A, we can easily express $H(\boldsymbol{I})$ in terms of the real eigenvalues $\lambda_j^I(j=1,..,N)$ of A as

$$H(\boldsymbol{I})/N = (1+\ln(2\pi)+\overline{\ln\lambda^I})/2, \quad (5)$$

where

$$\overline{\ln\lambda^I} \equiv \left(\sum_j \ln\lambda_j^I\right)/N. \quad (6)$$

$H(\boldsymbol{I})/N$ may be considered to represent an entropy production rate per time Δ for the input signal. If we denote the covariance matrix of the output $\boldsymbol{O}$ by B and also the covariance matrix of the $2N$ dimensional vector $(\boldsymbol{I},\boldsymbol{O})$ by Z, we have a simple expression for $M(\boldsymbol{I}:\boldsymbol{O})$ from (1) and (2) as

$$M(\boldsymbol{I}:\boldsymbol{O})/N = [\overline{\ln\lambda^I}+\overline{\ln\lambda^O}]/2 - \overline{\ln\lambda^{I,O}}, \quad (7)$$

where

$$\overline{\ln\lambda^{I,O}} \equiv (\sum_j \ln\lambda_j^{I,O})/(2N), \quad (8)$$

with $\lambda_j^{I,O}(j=1,..,2N)$ denoting the eigenvalues of the matrix Z.

The final step is to relate the eigenvalues to the power spectra of time correlation functions which gives the matrix elements of various covariance matrices introduced above. Since the elements $A_{j,k}$ of the matrix A depends on the difference $|j-k|$ only, we notice after Fourier transformation that the eigenvalues is given by $\lambda_j^I = \sum_k c_{I,I}(k)\exp(ik\omega_j) \equiv C_{I,I}(\omega_j)$ with ω_j taking the values $\omega_j = 2\pi j/N(j=0,\pm1,..\pm N/2)$. λ_j^O is similarly given in terms of the spectrum $C_{O,O}(\omega_j)$. As to the eigenvalues of the $2N\times 2N$ matrix Z, we notice, employing a similar Fourier transformation approach as for A, that the eigenvalue $\lambda(\omega)$ satisfies the linear equation

$$C_{I,I}(\omega)I(\omega) + C_{I,O}(\omega)O(\omega) = \lambda(\omega)I(\omega), \quad (9)$$

$$C_{O,I}(\omega)I(\omega) + C_{O,O}(\omega)O(\omega) = \lambda(\omega)O(\omega), \quad (10)$$

with $I(\omega) \equiv \sum_k i_k\exp(i\omega k)$ and similarly for $O(\omega)$. This situation is similar to the problem of eigenfrequency for a two-component harmonic lattice. Thus we have two branches

$$\lambda_\pm(\omega) = [C_{I,I}(\omega)+C_{O,O}(\omega)\pm\sqrt{D}]/2, \quad (11)$$

with $D = (C_{I,I}(\omega)-C_{O,O}(\omega))^2 + 4|C_{I,O}(\omega)|^2$.

Collecting the results above we have a simple integral formula for the mutual information rate $m(\boldsymbol{I}:\boldsymbol{O}) \equiv M_I(\boldsymbol{I}:\boldsymbol{O})/(N\Delta)$ of the form

$$m(\boldsymbol{I}:\boldsymbol{O}) = (1/2)\int_{-f_0}^{f_0} df[\ln(C_{I,I}(f)) + \ln(C_{O,O}(f) - \ln(\lambda_+(f)) - \ln(\lambda_-(f))] = (-1/2)\int_{-f_0}^{f_0} df \ln(1 - |C_{I,O}(f)|^2/[C_{I,I}(f)C_{O,O}(f)]), \quad (12)$$

where f_0 denotes the Nyquist frequency $(1/2\Delta)$ with $f = \omega/(2\pi)$. Here we used the relation $\lambda_+(f)\lambda_-(f) = C_{I,I}(f)\ C_{O,O}(f) - |C_{I,O}(f)|^2$. We first remark that $m(\boldsymbol{I}:\boldsymbol{O}) = 0$ if the cross correlation or cross spectrum is zero as it should be [8]. Secondly in the subthreshold SR, where the input signal is small compared with (Gaussian) noise as studied here, the Gaussian approximation seems to be a reasonable zeroth-order approximation [9] although it remains to be quantitatively checked. Final remark is about SR in biological systems [15,16]. (12) has been used under a linear approximation [16] $\ln(1-K) \simeq -K$ where $K \equiv |C_{I,O}(f)|^2/[C_{I,I}(f)C_{O,O}(f)]$. Under this approximation $m(\boldsymbol{I}:\boldsymbol{O})$ is linear in K. We will instead employ the full expression (12) based on detailed numerical experiments for the FHN system (13).

3 SR in a FitzHugh-Nagumo(FHN) neuron

As a concrete system to be studied with the DMI, (12), we take the FHN neuron, first studied in detail in relation to SR in [17], whose dynamics is described after some reduction by [7,9,11]

$$\epsilon dv/dt = -v(v^2 - 1/4) - w + a + S(t) + \xi(t), \quad dw/dt = v - w, \quad (13)$$

where $S(t)$ denotes the input signal and $\xi(t)$ the Gaussian white noise with

$$\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t'). \quad (14)$$

When the parameter a is larger than $a_T = -5/(12\sqrt{3}) \simeq -0.24$, (13) has a limit cycle solution in the absence of the input signal and noise. We are mainly interested in a subthreshold situation $a < a_T$ and take $a = -0.3$ in all our subsequent calculations and ϵ =0.005. $v(t)$ corresponds to a fast voltage variable and $w(t)$ to a slow recovery variable [7].

For SR to a periodic signal $S(t)$ [3,6], one often takes

$$S_p(t) = \gamma \sin(\omega_0 t). \quad (15)$$

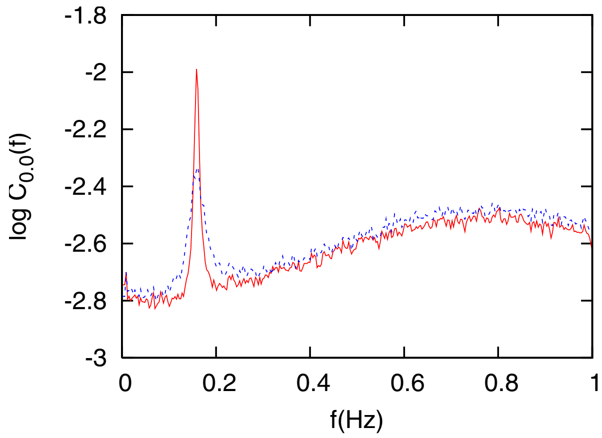


Fig. 1. Semi-log plot of the power spectrum $C_{O,O}(f)$ of the FHN model for two different η, $\eta = 0.05$ (a solid curve) and $\eta = 0.2$ (a dashed curve) ($D_s = 10^{-3}, f_0 = 2\pi, D = 1.5\times10^{-4}$ and $\gamma = 0.01$).

Since we take the DMI as a measure for SR, we consider the stochastic-periodic $\tilde{S}(t)$ described by the Langevin equation

$$d^2\tilde{S}/dt^2 = -\omega_0^2\tilde{S} - \eta d\tilde{S}/dt + \xi_s(t), \quad \langle \xi_s(t)\rangle = 0, \quad <\xi_s(t)\xi_s(t')\rangle = 2D_s\delta(t-t'). \quad (16)$$

In the limit $\eta \to 0$ and $D_s \to 0$, we recover the sinusoidal signal (15). Since the variance of the stationary Gaussian process $\tilde{S}(t)$ is given by $D_s/(\eta\omega_0^2)$, we may consider that the signal defined by

$$S(t) = C_0\tilde{S}(t), \qquad C_0 = \gamma\omega_0\sqrt{\eta/(2D_s)}, \quad (17)$$

has a similar amplitude to $S_p(t)$, (15), since the time average of the square of the input signal $S(t)$ becomes $\gamma^2/2$ as for the periodic signal, (15). Hereafter we set $\omega_0 = 1$.

For SR to aperiodic signals, we first introduce the Ornstein-Uhlenbeck process $\tilde{S}(t)$ described by

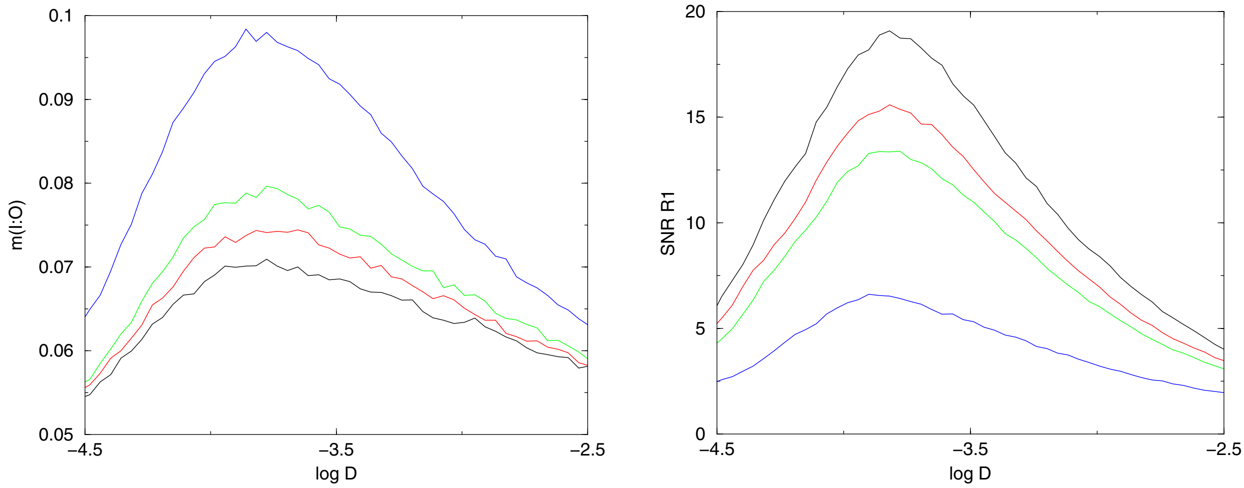
$$d\tilde{S}(t)/dt = -\eta\tilde{S}(t) + \xi_s(t), \quad (18)$$

with the noise $\xi_s(t)$ having the same properties as in (16). In this case we express , from a similar argument as above, our input signal to be

$$S(t) = C_1\tilde{S}(t), \qquad C_1 = \gamma\sqrt{\eta/(2D_s)}. \quad (19)$$

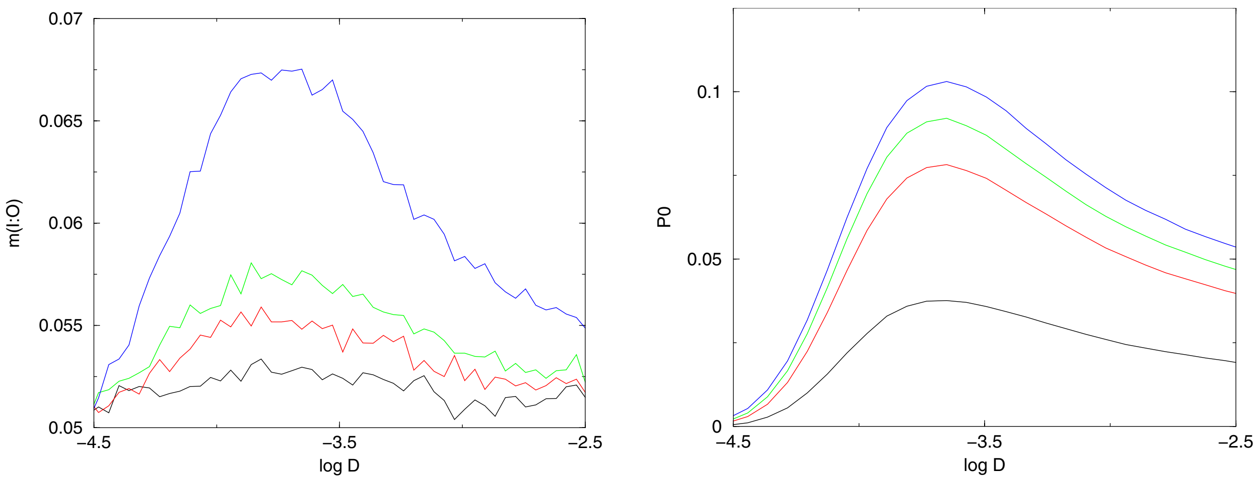
In normalizing the amplitude of the signal $S(t)$, we choose $\gamma = 0.01$, which ensures that our signal is small and subthreshold.

We consider first the case of a stochastic-periodic input (17) to the FHN model. In Figure 1 the power spectra of the output signal $V(t) \equiv \Theta[v(t)]$ with $\Theta(x) = 1(0)$ for $x > 0$ ($x \le 0$) are plotted for two different η, whose inverse measures coherence of the input signal $S(t)$. As expected the peak around $f = f_0$ is sharper for the smaller damping $\eta = 0.05$ compared with the larger damping $\eta = 0.2$. In this stochastic-periodic case, the SNR could be defined either by $R_1 \equiv C_{O,O}(f_0)/C_{O,O}^{NB}(f_0)$ [1], with $C_{O,O}^{NB}(f_0)$ denoting the noise background at $f = f_0$ of $C_{O,O}(f)$ or by



(a) (b)

Fig. 2. MI rate $m(\boldsymbol{I}:\boldsymbol{O})$(a) and SNR, R_1(b) as a function of $\log_{10} D$ for four different $\eta's$.($\eta = 0.001, 0.005, 0.01$ and 0.05 from below for Figure 2a and from above for Figure 2b. with $D_s = 10^{-3}$, $f_0 = 2\pi$ and $\gamma = 0.01$.



(a) (b)

Fig. 3. MI rate $m(\boldsymbol{I}:\boldsymbol{O})$ (3a. left) and the power norm $P_0 \times 10^3$, equation (20)(3b. right) as a function of $\log_{10} D$ for four different $\eta's$($\eta = 0.001$, 0.005, 0.01 and 0.05) from below for both 3a. and 3b. We put $D_s = 10^{-3}$ and $\gamma = 0.01$.

$R_2 \equiv C_{O,O}(f_0)/C_{O,O}(f_1)$ with f_1 denoting the frequency at which $C_{O,O}(f)$ takes its local minimum between $f = 0$ and f_0. As the third candidate for the SNR, we also considered $R_3 = A/C_{O,O}^{NB}(f_0)$ where A denotes the area of the (nearly)triangle region around f_0 above the background noise level.

In Figure 2a we show $m(\boldsymbol{I}:\boldsymbol{O})$, (12), as a function of the noise intensity D, (14), for four different values of η. We observe a peak of $m(\boldsymbol{I}:\boldsymbol{O})$ at some intermediate noise strength. Similar SR behavior is also observed in Figure 2b, which depicts the SNR R_1 as a function of the noise intensity D. It is noted that R_2 and R_3 show similar SR behavior as R_1.

One point is worth noting here. From Figure 2a it is seen that the larger the damping η is, the larger $m(\boldsymbol{I}:\boldsymbol{O})$ is and this tendency is opposite to the one shown in Figure 2b for the SNR. We interpret this as follows: The input signal carries its information in passing through the FHN neuron. When η is small, the signal is rather coherent and structureless and in this sense we may consider that it carries small amount of information. Consequently $m(\boldsymbol{I}:\boldsymbol{O})$ for a small η inevitably becomes relatively small compared with the case of larger η. In fact, the input entropy rate (5), which is simply expressed in terms of $C_{I,I}(f)$ as (12), has been calculated to see that it monotonically increases as a function of η. Thus if we consider a normalized quantity $\tilde{M}(\boldsymbol{I}:\boldsymbol{O}) \equiv M(\boldsymbol{I}:\boldsymbol{O})/H(\boldsymbol{O})$ instead of $M(\boldsymbol{I}:\boldsymbol{O})$, this behaves as a function of η like $R_1 \equiv C_{O,O}(f_0)/C_{O,O}^{NB}(f_0)$ [1], which is normalized by the noise background $C_{O,O}^{NB}(f_0)$.

Now we turn to aperiodic SR where the input signal is produced by (18) and (19). In Figure 3a is plotted $m(\boldsymbol{I}:\boldsymbol{O})$ as a function of D. Here we also observe systematic behavior of $m(\boldsymbol{I}:\boldsymbol{O})$ as in the case of a stochastic-periodic SR. That is, from a MI viewpoint there exists SR for the FHN model. Heretofore aperiodic SR has been studied mainly based on a measure called the power norm P_0 defined by [7]

$$P_0 \equiv \langle S(t)(R(t) - \overline{R(t)})\rangle, \tag{20}$$

where $R(t)$ is the mean firing rate constructed from the signal $v(t)$ and $\langle ... \rangle$ and the overbar denotes an ensemble average and time average, respectively. We show P_0 in Figure 3b and notice that P_0 also show its peak around a similar noise intensity. Since P_0 is, like $M(\boldsymbol{I} : \boldsymbol{O})$, not normalized by a quantity related to noise or input signal intensity, we have no inversion ordering between $m(\boldsymbol{I} : \boldsymbol{O})$ and P_0 with respect to η.

4 Summary

In this report we discussed SR in the FHN neuron model, (13), based on the DMI under a Gaussian approximation, (12). The DMI enabled us to consider SR for both (stochastic) periodic input signals, (16), and aperiodic input signals, (18), on equal footing. As to SR to periodic(or aperiodic) input signals, DMI turned out to be a good measure to estimate performance of the FHN model like the SNR R_1(or the power norm P_0).

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